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EXPLICIT BASIS FOR THE IRREDUCIBLE REPRESENTATION OF THE GROUP O(2, 1) IN THE TERMS OF EIGENFUNCTIONS OF THE TWO DIMENSIONAL PSEUDO-COULOMB PROBLEM

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ABSTRACT

We determine explicitly (including the phase) the ortonormalized eigenstates of the pseudo-Coulomb problem that are basis for the irreducible representation of its symmetry group O(2,1). Furthermore, we have an explicit realization of the algebra o(2,1) in terms of diferential operators, operating on the Hilbert Space $L^2(R^2)$.

I. INTRODUCTION

It is well known that the Coulomb problem with a two dimensional repulsive potential

$$\left(\frac{1}{2}P^2 - R^{-1}\right)\psi = (2\nu^2)^{-1}\psi \quad , \tag{I.1}$$

with ν, any real number, could be transformed through the dilatation $\underline{\rho} = \nu^{-1}\underline{R}$, $\underline{\pi} = \nu\underline{P}$ to the pseudo-Coulomb problem, whose Schrodinger equation, in polar coordinates, is

$$\frac{1}{2}\rho\left(-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} - \frac{1}{\rho}\frac{\partial^2}{\partial\theta^2} - 1\right)\psi = \nu\psi \quad . \tag{I.2}$$

In ref. (1), it is shown, in abstract form, that the eigenstates of definite angular momentum $|\nu,m\rangle$ of the problem (I.2) are basis of an

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irreducible representation of O(2,1); that is, the generators of O(2,1) denoted by T_+, T_-, T acting on the basis, give us

$$T_{\pm} |\nu,m> = \sqrt{(\nu^2 + 1/4) + m(m \pm 1)}\ |\nu,m \pm 1> \tag{I.3}$$

and

$$T_3 |\nu,m> = m|\nu,m> \quad , \tag{I.4}$$

T_3 coincides with the angular momentum and m is an integer.

In this work, the basis of the representation, is explicitly determined. For this purpose, in the next section we solve (I.2), wich gives us the eigenfunctions of the pseudo-Coulomb problem in terms of Whittaker functions. These eigenfunctions are normalized up to a phase, which is determined using (I.3) and well known properties of the Whittaker functions.

II. PSEUDO-COULOMB EIGENFUNCTIONS

In order to find the pseudo-Coulomb eigenstates, we must solve the equation (I.2). Taking angular momentum eigenstates, we can write the states in the following form

$$\psi_\nu^m (\rho,\theta) = R_{\nu,m}(\rho)\ e^{im\theta} \ . \tag{II.1}$$

If we consider the radial functions as

$$R_{\nu,m}(\rho) = \frac{f_{\nu,m}(\rho)}{\sqrt{\rho}} \quad , \tag{II.2}$$

it follows from (I.2) that $f_{\nu,m}(\rho)$ satisfies the equation

$$\frac{d^2 f_{\nu,m}}{d\rho^2} + \left(1 + \frac{2\nu}{\rho} + \frac{\frac{1}{4} - m^2}{\rho^2} \right) f_{\nu,m} = 0 \tag{II.3}$$

This is Whittaker's equation in the variable $2i\rho$, whose solution is given by[2]

$$f_{\nu,m}(\rho) = M_{-i\nu,|m|}(2i\rho) \tag{II.4}$$

with

$$M_{\lambda,\mu}(z) = z^{\mu+1/2}\, e^{-z/2}\ {}_1F_1(\mu - \lambda + \tfrac{1}{2},\ 2\mu + 1,\ z) \qquad \text{(II.5)}$$

The second linearly independent solution of (II.3), denoted by $M_{\lambda,-\mu}$ and of the same form as (II.5) with the change $\mu\to-\mu$, diverges at the origen and we don't considered it. The complet set of the pseudo-Coulomb eigenstates[3] is

$$|\nu,m\rangle = B_{\nu,m}\ \frac{M_{-i\nu,\,|m|}(2i\rho)}{\sqrt{\rho}}\ e^{im\theta}\ , \qquad \text{(II.6)}$$

with ν any real, m integer and $B_{\nu,m}$ is the normalization constant, once we normalize the states through the condition

$$\langle\nu',m'|\nu,m\rangle = \delta(\nu - \nu')\ \delta_{mm'}\ . \qquad \text{(II.7)}$$

The measure is given by $\frac{1}{2}\,d\rho\, d\theta$. The choice of this measure follows from the hermiticity conditions for the pseudo-Coulomb Hamiltonian and the group generators. Putting the eigenfunctions (II.6) into the normalization conditions (II.7), gives us

$$\langle\nu',m'|\nu,m\rangle = \frac{B^*_{\nu',m'}B_{\nu,m}}{2}\int_0^\infty\!\!\int_0^{2\pi} \frac{M^*_{-i\nu',\,|m'|}(2i\rho)M_{-i\nu,\,|m|}(2i\rho)}{\sqrt{\rho}\ \sqrt{\rho}}\ e^{i(m-m')\theta}\, d\rho\, d\theta\ , \qquad \text{(II.8)}$$

which reduces immediatly to

$$\langle\nu',m'|\nu,m\rangle = \pi\, B^*_{\nu',m'}B_{\nu,m}\ \delta_{mm'}\int_0^\infty \frac{d\rho}{\rho}\ M^*_{-i\nu',\,|m|}(2i\rho)M_{-i\nu,\,|m|}(2i\rho) \qquad \text{(II.9)}$$

by performing the θ integral. If we use the following integral representation of the Whittaker functions[2]

$$M_{-i\nu,\,|m|}(2i\rho) = \frac{2^{i\nu+1}(2|m|)!\sqrt{\rho}\ e^{-\nu\pi/2}e^{i\rho}}{\Gamma\,(-i\nu + |m| + 1/2)}\int_0^\infty x^{-2i\nu}\ e^{i\,x^2/2}\ J_{2|m|}(2x\sqrt{\rho})dx\ , \qquad \text{(II.10)}$$

the equation (II.9) takes the form

$$\langle\nu',m'|\nu,m\rangle = \pi\ \delta_{mm'}B^*_{\nu',m}\ B_{\nu,m}\ \frac{2^{i(\nu-\nu')+2}((2|m|)!)^2\ e^{-\pi(\nu+\nu')/2}}{\Gamma(|m| + \frac{1}{2} - i\nu)\,\Gamma(|m| + \frac{1}{2} + i\nu')}$$

$$\int_0^\infty dy\ y^{2i\nu'}\ e^{-iy^2/2}\int_0^\infty dx\ x^{-2i\nu}e^{ix^2/2}\int_0^\infty J_{2|m|}(2y\sqrt{\rho})\ J_{2|m|}(2x\sqrt{\rho})d\rho\ . \qquad \text{(II.11)}$$

Taking into account[4]

$$\int_0^\infty J_{2|m|}(2y\sqrt{\rho})\; J_{2|m|}(2x\sqrt{\rho})\; d\rho \;=\; \frac{1}{2y}\,\delta(x-y) \quad , \qquad \text{(II.12)}$$

the integral simplifies with the use of Dirac's delta, giving

$$\langle\nu',m'|\nu,m\rangle = \pi\,\delta_{mm'}\,B^*_{\nu',m}B_{\nu,m}\,\frac{2^{i(\nu-\nu')+1}\,((2|m|)!)^2\, e^{-\pi(\nu+\nu')/2}}{\Gamma(|m|+\frac{1}{2}-i\nu)\,\Gamma(|m|+\frac{1}{2}+i\nu')}\int_0^\infty \frac{dy}{y}\; y^{2i(\nu-\nu)} \qquad \text{(II.13)}$$

The last integral is evaluated with the change $t = \ln y$, obtaining

$$\int_0^\infty \frac{dy}{y}\; y^{2i(\nu-\nu')} \;=\; \pi\;\delta(\nu-\nu')\;. \qquad \text{(II.14)}$$

From (II.7), (II.13) and (II.14) we find that the normalization constant is

$$B_{\nu,m} \;=\; e^{i\delta(\nu,m)}\sqrt{\frac{\Gamma(|m|+\frac{1}{2}-i\nu)\,\Gamma(|m|+\frac{1}{2}+i\nu)}{\pi\sqrt{2}\;(2|m|)!}}\; e^{\pi\nu/2}\;, \qquad \text{(II.15)}$$

where $\delta(\nu,m)$ is a phase.

III. DETERMINATION OF THE PHASE

Taking into account the recursion relations of the Whittaker functions, we immediatly obtain the following formulas for the radial part of the wave function (II.1):

$$\mathcal{D}_{\pm}(\nu,|m|)\;R_{\nu,|m|} \;=\; C_{\pm}(\nu,|m|)\;R_{\nu,|m|\pm 1} \qquad m \neq 0$$

$$\mathcal{D}_{\pm}(\nu,|m|)\;R_{\nu,0} \;=\; C_{+}\;(\nu,0)\;R_{\nu,1} \quad , \qquad \text{(III.1)}$$

where

$$\mathcal{D}_{\pm}\;(\nu,|m|) \;=\; -\left(\nu - \frac{|m|\,(|m|\,\pm\frac{1}{2})}{\rho} \;\pm\; (|m|\,\pm\frac{1}{2})\,\frac{d}{d\rho}\right)\;, \qquad \text{(III.2)}$$

$$C_+ (\nu,|m|) = -i \frac{(|m| + \frac{1}{2}) + \nu^2}{4(|m| + \frac{1}{2})(|m| + 1)} ,$$

$$C_- (\nu,|m|) = 4i \; |m| \; (|m| - \frac{1}{2}) \; . \qquad \text{(III.3)}$$

From reference (1) it is deduced that the O(2,1) generators, in polar coordinates, take the form

$$T_{\pm} = -e^{\pm i\theta} \{ H + \frac{1}{\rho} \frac{\partial^2}{\partial\theta^2} \pm \frac{i}{2\rho} \frac{\partial}{\partial\theta} + \frac{1}{2} \frac{\partial}{\partial\rho} \mp i \frac{\partial^2}{\partial\rho\partial\theta} \} \qquad \text{(III.4)}$$

$$T_3 = -i \frac{\partial}{\partial\theta} \qquad \text{(III.5)}$$

where H is the pseudo-Coulomb Hamiltonian and T_3 coincides with the angular momentum operator. Applying $T_{\pm}$ to the pseudo-Coulomb eigenstates and making use of (I.2) we obtain

$$T_{\pm} \; |\nu,m\rangle = B_{\nu,m} \; (\mathcal{T}_{\pm} \; R_{\nu,|m|}) \; e^{i(m \pm 1)\theta} \; , \qquad \text{(III.6)}$$

where

$$\mathcal{T}_{\pm} \; (\nu,m) = - \left(\nu - \frac{m(m \pm \frac{1}{2})}{\rho} \pm (m \pm \frac{1}{2}) \frac{d}{d\rho} \right) \qquad \text{(III.7)}$$

The difference between this operator and $\mathcal{D}_{\pm}(\nu,|m|)$ rest in the fact that this last is defined for positive values of m. We note that

$$\mathcal{T}_{\pm}(\nu,m) = \begin{cases} \mathcal{D}_{\pm} \; (\nu,|m|) & m \geq 0 \\ \mathcal{D}_{\pm} \; (\nu,|m|) & m < 0 \; , \end{cases} \qquad \text{(III.8)}$$

thus

$$T_{\pm}|\nu,m\rangle = B_{\nu,m} \begin{cases} \mathcal{D}_{\pm}(\nu,|m|) \; R_{\nu,|m|} \; e^{i(m\pm 1)\theta} & m \geq 0 \\ \mathcal{D}_{\mp}(\nu,|m|) \; R_{\nu,|m|} \; e^{i(m\pm 1)\theta} & m < 0 \; . \end{cases} \qquad \text{(III.9)}$$

Utilizing (III.1) and after a small computation we obtain

$$
T_{\pm}|\nu,m\rangle = \begin{cases} \left(\dfrac{B_{\nu,|\mu|}}{B_{\nu,|\mu\pm 1|}} \; C_{\pm}(\nu,|\mu|) \right)_{\mu=m} |\nu, m\pm 1\rangle & m \geq 0 \\ \\ \left(\dfrac{B_{\nu,-|\mu|}}{B_{\nu,-|\mu\pm 1|}} \; C_{\mp}(\nu,|\mu|) \right)_{\mu=m} |\nu, m\pm 1\rangle & m < 0 \end{cases} \qquad \text{(III.10)}
$$

If we compare this equation and (I.3), we find a unique recursion formula for the coefiecients $B_{\nu,m}$, given by

$$
B_{\nu,\pm(|m|+1)} = -i \frac{\sqrt{\nu^2 + (|m| + \frac{1}{2})^2}}{4(|m| + \frac{1}{2})(|m| + 1)} \; B_{\nu,\pm|m|} \quad . \qquad \text{(III.11)}
$$

For this equation and (II.15) we have the relation between the -- phases

$$
\delta(\nu, \pm|m| + 1) = -\frac{\pi}{2} + \delta(\nu, \pm|m|) \quad , \qquad \text{(III.12)}
$$

and under the condition that the phase for zero angular momentum is zero, we have

$$
\delta(\nu,m) = -\frac{\pi m}{2} \quad . \qquad \text{(III.13)}
$$

IV. CONCLUSION

We have shown that the eigenstates for the pseudo-Coulomb problem, given explicitly by

$$
|\nu,m\rangle = e^{\pi(\nu - im)/2} \frac{\sqrt{\Gamma(|m| + \frac{1}{2} - i\nu)\Gamma(|m| + \frac{1}{2} + i\nu)}}{\pi\sqrt{2}\,(2|m|)!} \frac{M_{-i\nu,|m|}(2i\rho)}{\sqrt{\rho}} e^{im\theta} \qquad \text{(IV.1)}
$$

form a basis for an irreductible representation of the Lie algebra o(2,1) of the symetry group O(2,1) of the problem. Furthermore, we have given an explicit realization of the algebra o(2,1) in terms of formal diiferential operators, operating on the Hilbert Space $L^2(R^2)$.

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